

# Approximation Power of Directionlets

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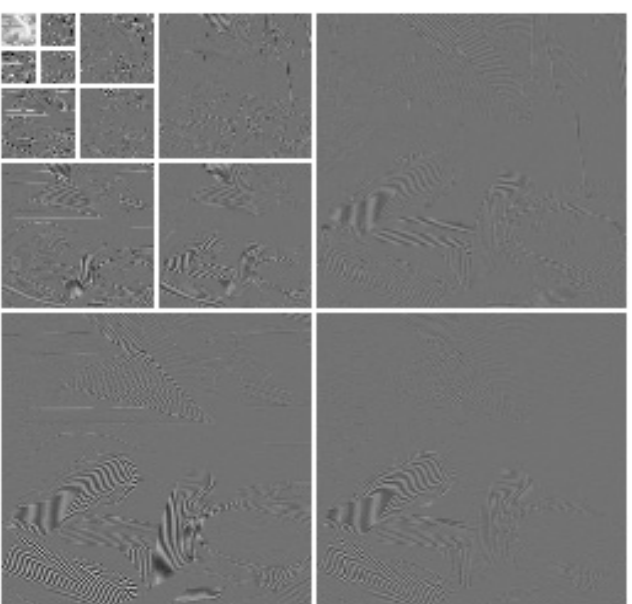
<http://lcavwww.epfl.ch/~vladan,beferull,vetterli>

<http://www.commsp.ee.ic.ac.uk/~pld>

# Outline

- **Introduction**
- Anisotropic transforms
- Skewed transforms
- Applications in image processing:
  - Non-linear approximation
  - Compression
  - Denoising

## Standard 2-D Wavelet Transform - Review



**Advantages:** sparse representation, separability, simple filter design (1-D), low computational complexity - used in JPEG2000

**Disadvantages:** isotropy, only horizontal and vertical directions

## Previous Work & Motivation

Several approaches with anisotropy or multi-directionality have been proposed:

- *Curvelets* (Candès & Donoho),
- *Contourlets* (Do et al.),
- *Wedgelets* (Donoho, Baraniuk),
- *Wedgeprints* (Wakin et al.),
- *Edgeprints* (Dragotti & Vetterli),
- *Bandelets* (Mallat),
- *Multiscale transform* (Cohen),
- *Directional wavelets* (Zuidwijk),
- *Polynomial modeling & quadtree segmentation* (Shukla et al.),
- *brushlets* (Meyer & Coifman)
- *Steerable pyramid* (Simoncelli),
- *Directional filter banks* (Bamberger & Smith),
- *Complex wavelets* (Kingsbury).

**Disadvantages:** non-separability (2-D convolution), continuous-time filter design, high computational complexity, oversampling, non-perfect reconstruction

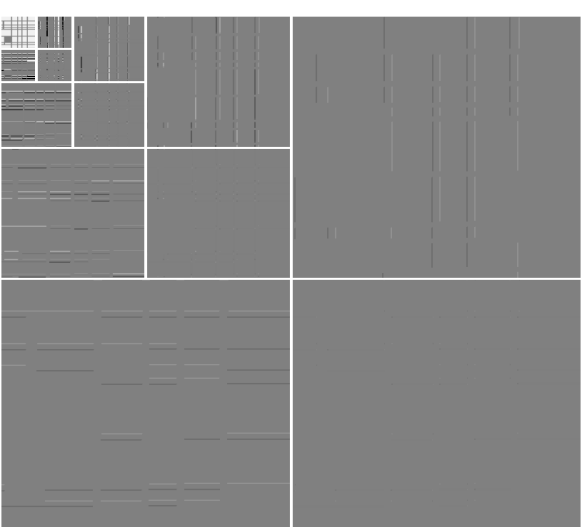
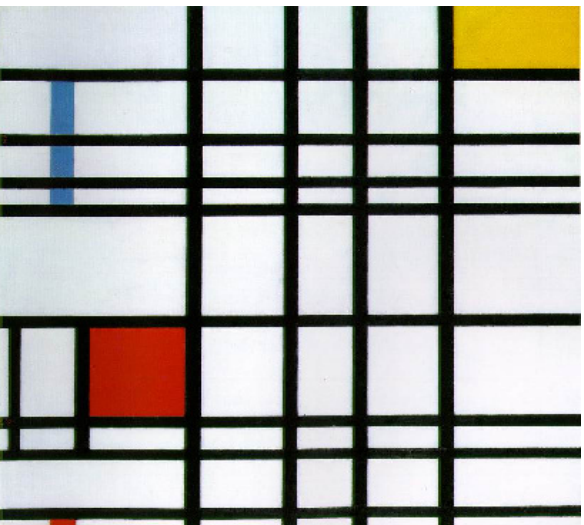
**Our goals:** anisotropy, multi-directionality, bases property, separable filter design

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# Mondrian Images

Standard 2-D wavelet transform of Mondrian images is not sparse



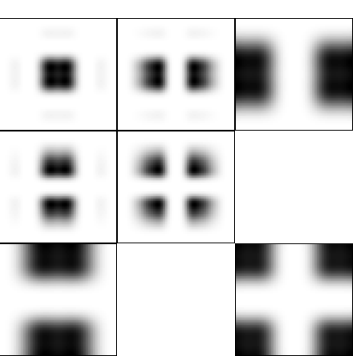
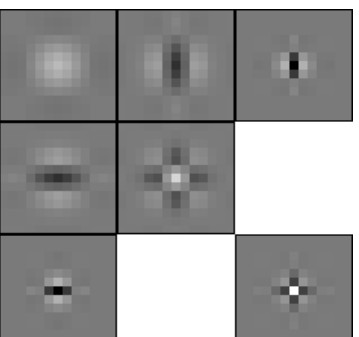
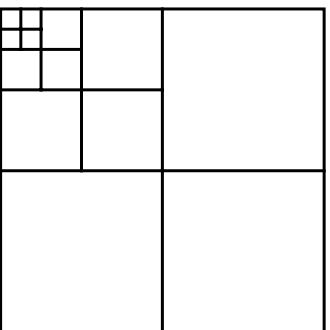
There are  $O((k_1 + k_2)M)$  nonzero coefficients

# Fully Separable Decomposition

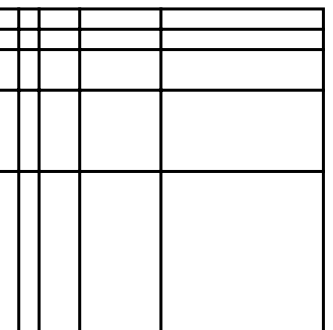
Tensor wavelets - previously proposed by Westerink (1989), Nowak & Baraniuk (1999), Rosiene & Nguyen (1999)

Biorthogonal "9-7" 1-D filters are used

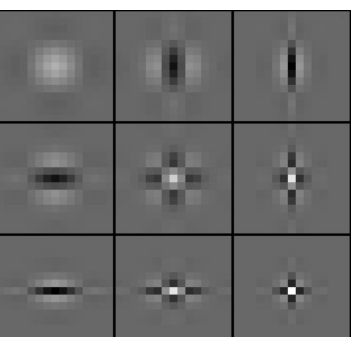
Stand.  
WT



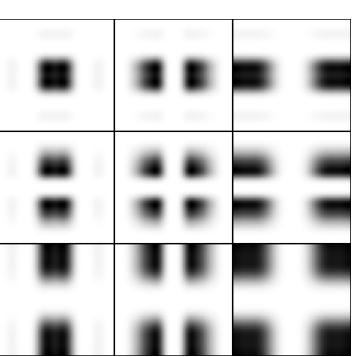
FSWT



Freq. decomp.



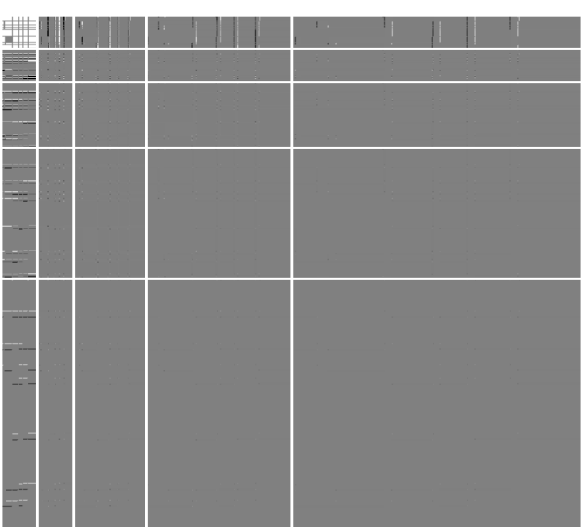
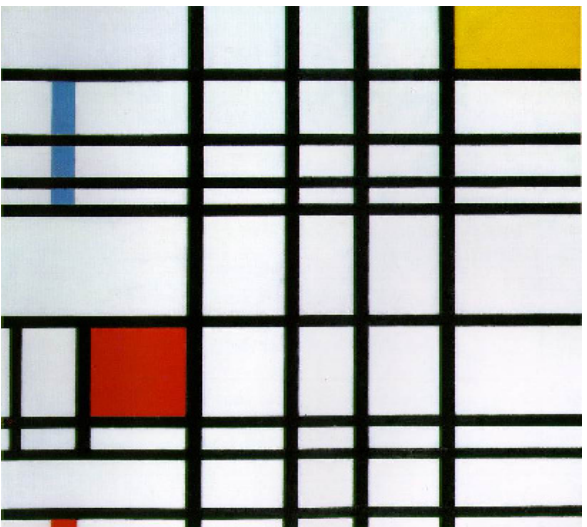
Imp. resp.



Spectrum

## Fully Separable Decomposition - Approximation Order

Fully separable transform changes the order of approximation of Mondrian images



The representation is sparser: there are  $O((k_1 + k_2)(\log_2 M)^2)$  nonzero coefficients

Gain over standard WT:  $(\log_2 M)^2 / M$

If horizontal and vertical directions dominate the result is fine.

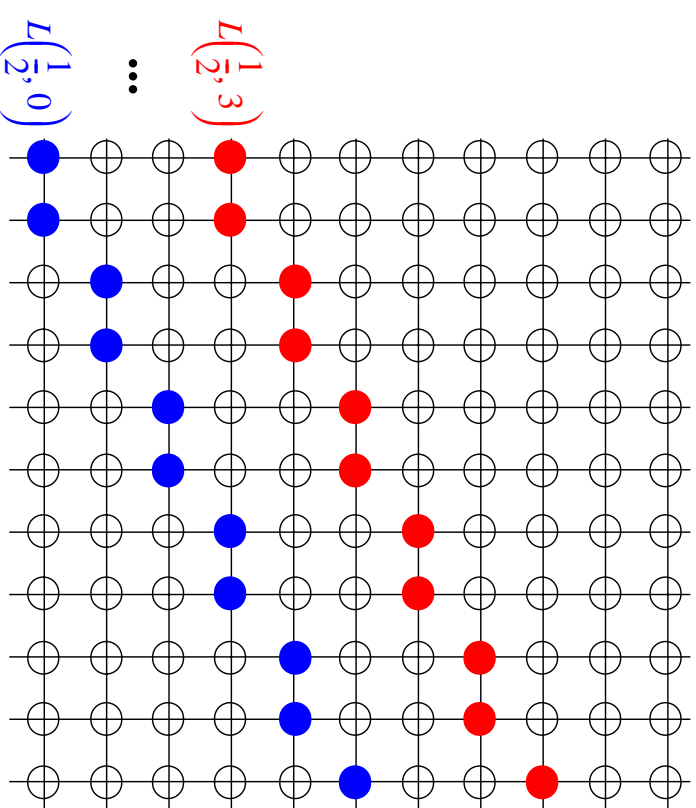
Otherwise, do it in other directions

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## Discretization of Directions

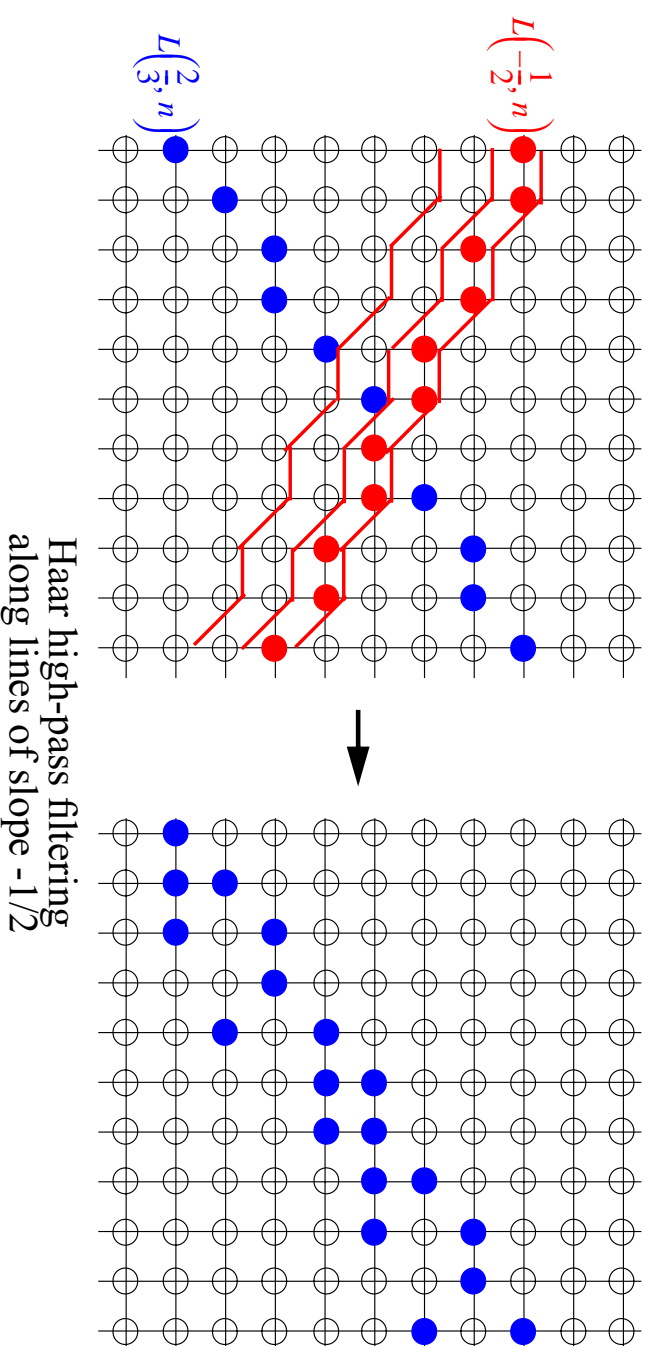
Digital line  $L(r, n)$  is approximation to the continuous line along the slope  $r$  and with the intercept  $n$  (Bresenham, 1965)



Digital lines  $L(r, n)$ , for a fixed rational  $r$  and  $n \in \mathbb{Z}$ , partitions the discrete space  $\mathbb{Z}^2$

## Directional Interaction

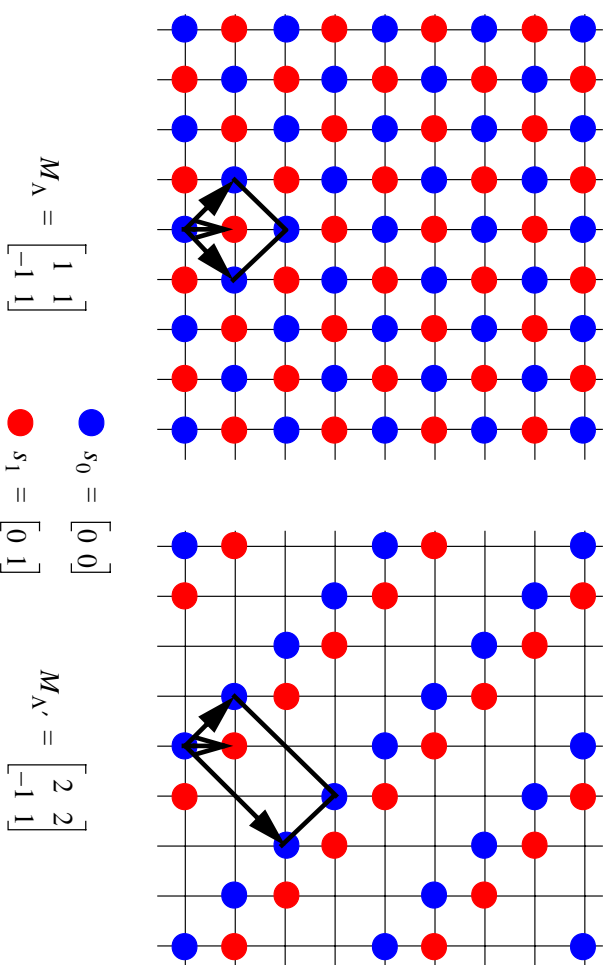
Apply filtering and subsampling along digital lines for a fixed direction  
Nonzero coefficients produced by one digital line are not aligned  
after filtering and subsampling!



# Lattice-based Filtering and Subsampling

Filtering and subsampling are applied across integer lattice  $\Lambda$  yielding  $\Lambda'$ :

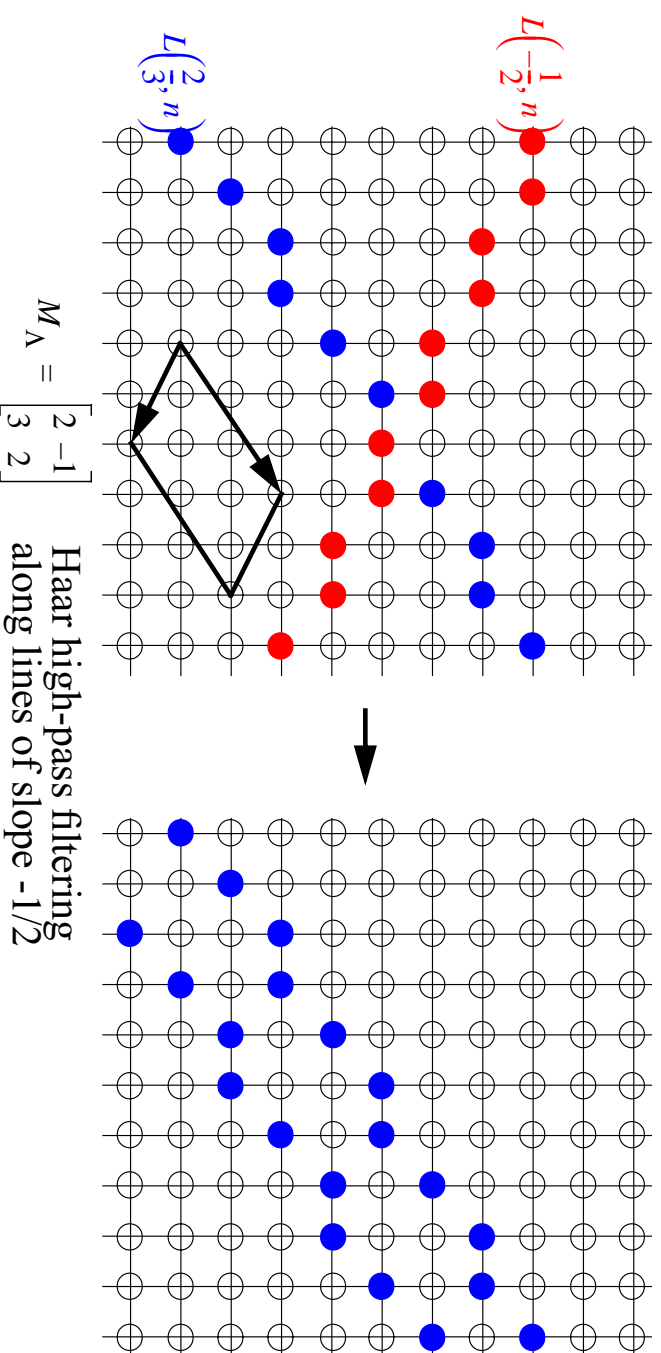
$$M_{\Lambda} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \longrightarrow M_{\Lambda'} = \begin{bmatrix} 2a_1 & 2b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 2d_1 \\ d_2 \end{bmatrix}$$



Operations are applied in each coset separately

# How to Avoid Directional Interaction

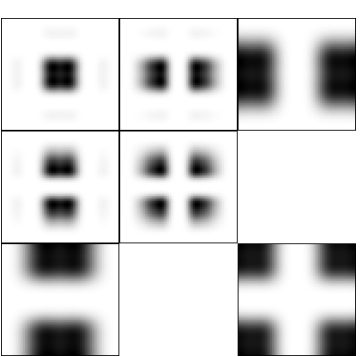
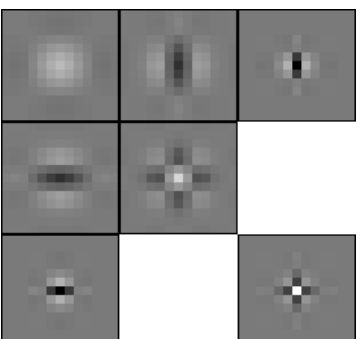
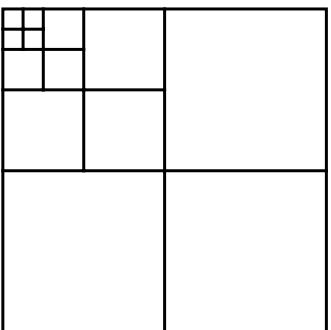
Nonzero coefficients produced by one digital line are aligned after filtering applied in lattices



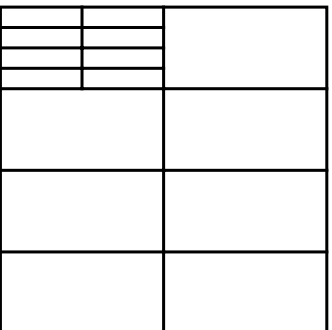
# Skewed Anisotropic Wavelet Transforms

Apply anisotropic transforms along vectors of the lattice  $\Lambda \longrightarrow$  S-AWWT( $M_{\Lambda, n_1, n_2}$ )  
 Basis functions (**directionlets**) have directional vanishing moments

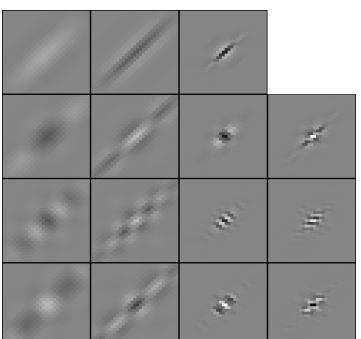
Stand.  
W/T



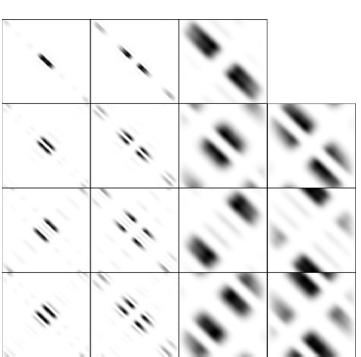
S-AWWT



Freq. decomp.



Imp. resp.



Spectrum

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## Approximation Power

Non-linear approximation: retain  $N$  largest magnitude transform coefficients

For  $C^2/C^2$  images mean-square error decays as:

- Standard WT:  $O(N^{-1})$  (Mallat)
- Optimal:  $O(N^{-2})$  (Mallat)
- **Oversampled** curvelets (Donoho) and contourlets (Do&Vetterli):  $O(N^{-2}(\log N)^3)$
- **Oversampled** bandelets (Le Pennec&Mallat) and wedgelets (Donoho, Baraniuk):  $O(N^{-2})$

## Approximation Power - Theorem

**Theorem:** Given a  $C^2/C^2$  image and using spatial segmentation directionlets can achieve

- (a)  $\text{MSE} = O(N^{-1.55})$  in non-linear approximation and
- (b)  $D = O(R^{-1.55})$  in compression.

Transform and alignment directions are adapted to the content in each spatial segment. Adaptivity allows for a sparser representation.

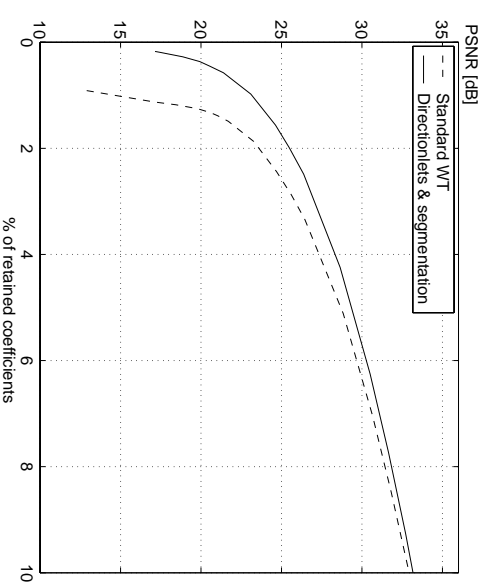
Directionlets are **critically sampled**  $\rightarrow$  convenient for Lagrangian optimization-based compression methods because mean-square error is conserved

# Approximation Results



Stand. WT

$N = 0.98\%$ ,  $MSE=13.93dB$



Optimized

$N = 0.98\%$ ,  $MSE=23.09dB$

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## Directionlets in Compression

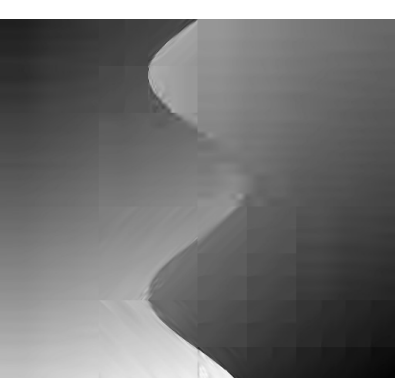
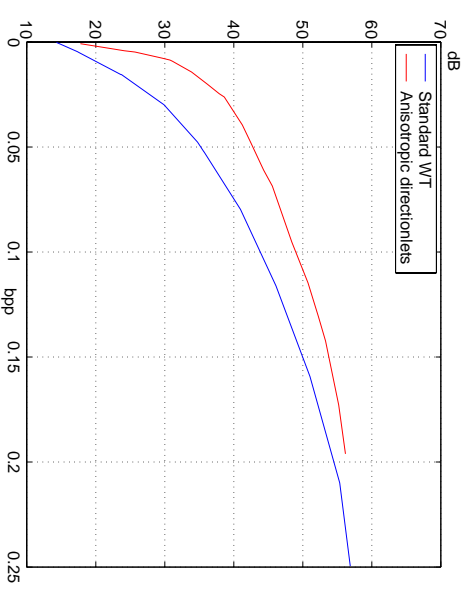
2-D wavelet transform has been applied successfully in image compression algorithms:

- Scalar quantization and optimal bit allocation (Antonini et al., 1992)
- Embedded coding using zerotrees (Shapiro, 1993)
- Set partitioning in hierarchical trees (SPIHT) (Said and Pearlman, 1996)
- Space-frequency quantization (Xiong, 1998)

**Sparsity** → **good compression!**

- Directionlets substitute the standard WT allowing for multi-directionality and anisotropy to improve sparsity
- Spatial segmentation extracts one or a few dominant directions per segment
- Given a target bitrate, the best segmentation and transform and alignment directions in each segment are found using Lagrangian optimization

## Preliminary Compression Results - Synthetic $C^2/C^2$ Images

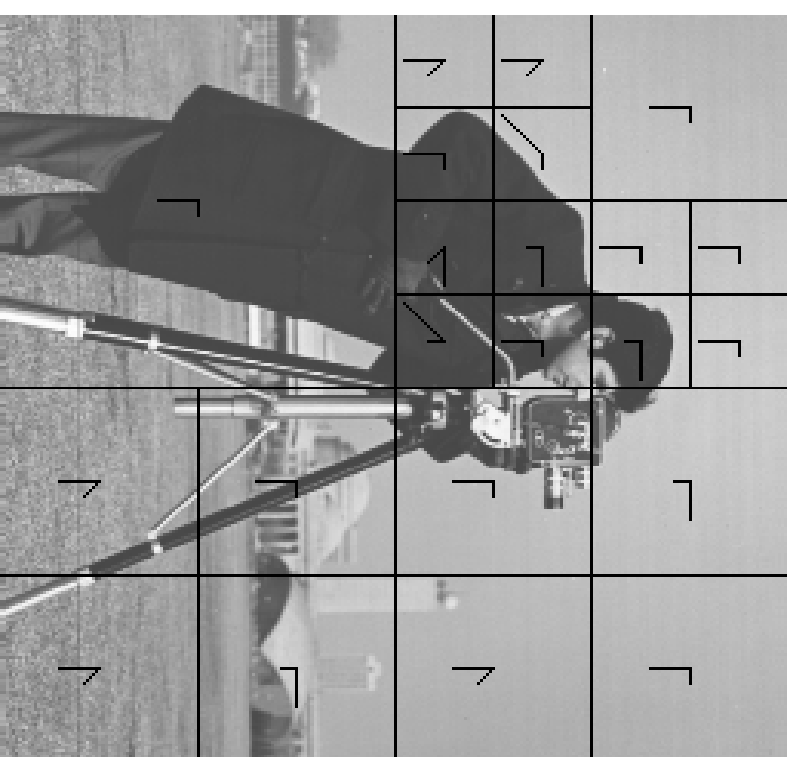
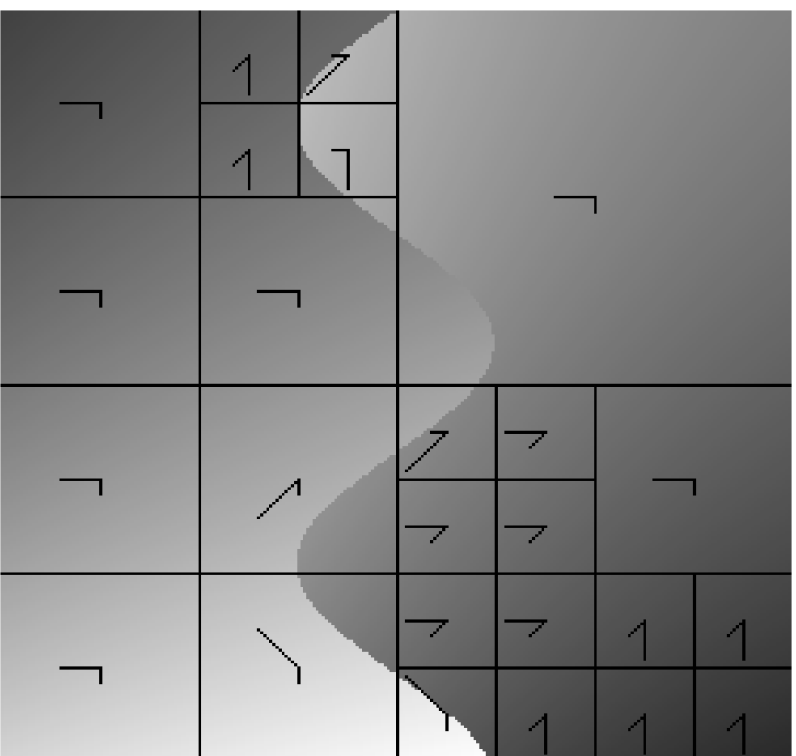


Stand. WT (0.0266bpp, 26.3dB)

Optimized (0.0266bpp, 38.6dB)

Compression of natural images is an on-going project

# Optimal Segmentation



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## Denoising Results

Redundant undecimated directionlets with Gaussian-scale mixtures  
(Wainwright et al. 2000, 2001; Portilla et al. 2003)

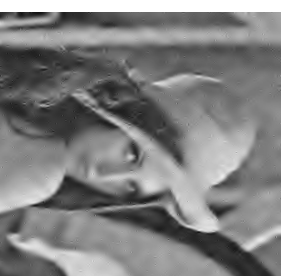
Enforce coherence across scales, space and directions in images



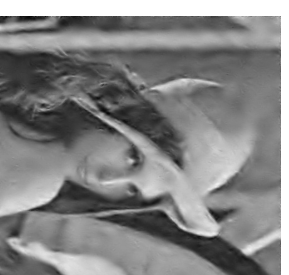
Original



Noisy 10.6dB



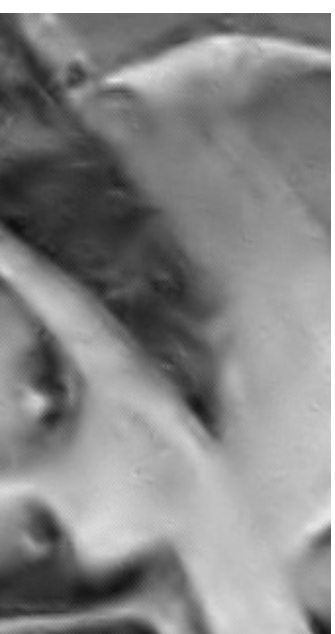
UWT 25.1dB



Dir-lets 26.1dB



UWT



Directionlets

## Conclusion

- Anisotropic wavelet transforms
- Lattice-based skewed wavelet transforms
- S-AWT (directionlets)
  - retains simplicity and separability of standard WT
  - involves multi-directionality and anisotropy
- Achievable order of approximation  $MSE = O(N^{-1.55})$ , while keeping critical sampling
- Straightforward implementation in compression with skewed anisotropic zerotrees
- Undecimated directionlets provide an efficient denoising tool enforcing coherence across scales, space and directions

Publications available at <http://lca-www.epfl.ch/~vladan/publications>